Philosophy 240: Symbolic Logic Hamilton College
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Class 6: More Truth Functions (§6.2)

I. **Determining the truth values of complex propositions, when one component is unknown**

We have seen how to calculate the truth value of a complex, or compound, proposition when the truth values of the components are known.
This property of classical logic, that truth values of long expressions are always computable from the truth values of simpler expressions, is called compositionality.
As I mentioned in the talk on conditionals, we lose compositionality if we don't treat the conditional truth-functionally (i.e. if we leave the values of the third and fourth row of the truth table blank.)

Sometimes you don't know truth values of one or more component variable.
(Soon we will dispense with the pretense that we know truth values of any un-interpreted letters.)
For purposes of this lesson, suppose that A, B, C are true; X, Y, Z are false; and P and Q are unknown.

Consider: $P \cdot A$
If P is true, then we have:
$T \cdot T$
which is true.
If P is false, then we have
$F \cdot T$
which is false.
Since the truth value of the compound expression depends on the truth value of P, it too is unknown.

But consider: $P \vee A$
If P is true, then we have
$T \vee T$
which is true.
If P is false, then we have
$F \vee T$
which is also true.
Since the truth value of the complex proposition is true in both cases, the value of that statement is true.

Similarly, consider: $Q \cdot Y$
If Q is true, then we have
$T \cdot F$
which is false.
If Q is false, then we have
$F \cdot F$
which is also false.
Since the truth value of the complex proposition is false in both cases, the value of that statement is false.
If the truth values come out the same in each case, then the statement has that truth value.
If the values come out differently in different cases, then the truth value of the statement is unknown.

II. **Exercises A**. Evaluate the truth value of each complex expression, using the same truth values as above.

1. $\sim(P \cdot X) \supset Y$
2. $P \supset A$
3. $A \supset P$
4. $Q \vee \sim Z$
5. $P \cdot \sim P$
6. $Q \vee \sim Q$
7. $\sim P \vee (\sim X \vee P)$
8. $[(P \supset X) \supset P] \supset P$
9. $(X \supset Q) \supset X$

III. **Determining the truth values of complex propositions, when more than one component is unknown**

Lastly, one can have more than one unknown in a statement. If there are two unknowns, we must consider four cases.

Consider: $\sim(P \cdot Q) \vee P$
If P and Q are both true
$\sim(T \cdot T) \vee T$
which is true.
If P is true and Q is false
$\sim(T \cdot F) \vee T$
which is true.
If P is false and Q is true
$\sim(F \cdot T) \vee F$
which is true.
If P and Q are both false
$\sim(F \cdot F) \vee F$
which is again true.

Since all possible substitutions of truth values yield a true statement, the statement is true.

IV. **Exercises B**. Evaluate the truth value of each complex expression, using the same truth values as above.

1. $(P \cdot Q) \vee (\sim Q \vee \sim P)$
2. $(P \vee Q) \cdot (\sim B \vee Y)$
3. $(P \supset Q) \supset \{[P \supset (Q \supset A)] \supset (P \supset A)\}$

V. **What is a truth table?**

When we are given a complex proposition, and we know the truth values of the component propositions, we can calculate the truth value of the longer statement.
When we are given a complex proposition, and at least some of the truth values of the component propositions are unknown, the best we can do, at times, is describe how the truth value of the whole varies with the truth value of its parts.

A truth table is a method which can help us characterize any logically complex proposition on the basis of the truth conditions of its component propositions.
Truth tables show us the distributions of all possible truth values of component propositions.
We can construct truth tables for any proposition, using the basic truth tables.
We can also use them to separate valid from invalid arguments.

VI. **Constructing truth tables for propositions**

The Method:

Step 1. How many rows do we need?
- 1 variable: 2 rows
- 2 variables: 4 rows
- 3 variables: 8 rows
- 4 variables: 16 rows
- n varialbes: 2^n rows

Step 2. Assign truth values to the component variables.
- We start truth tables always in the same ways.
- See below for examples.

Step 3. Work inside out, placing the column for each letter or connective directly beneath the letter or connective, until you complete the column under the main connective.

Examples:

For an example of a two-row truth table, consider the truth table for ‘P ⊃ P’

P	⊃	P
T	**T**	T
F	**T**	F

For an example of a two-row truth table, consider: ‘(P ∨ ~Q) · (Q ⊃ P)’
Step 1: We have two variables, so we need four rows.
Step 2: Assign truth values to component variables:

(P	∨	~	Q)	·	(Q	⊃	P)
T			T		T		T
T			F		F		T
F			T		T		F
F			F		F		F

Note that the same values we assign to P in the first column, we also use for P in the last column, and similarly for Q.
Also, all four row truth tables begin with this set of assignments.

Step 3, in stages:
First do the negation:

(P	∨	~	Q)	·	(Q	⊃	P)
T		**F**	T		T		T
T		**T**	F		F		T
F		**F**	T		T		F
F		**T**	F		F		F

Then the disjunction and conditional:

(P	∨	~	Q)	·	(Q	⊃	P)
T	**T**	F	T		T	**T**	T
T	**T**	T	F		F	**T**	T
F	**F**	F	T		T	**F**	F
F	**T**	T	F		F	**T**	F

And finally, the main connective, the conjunction, using the columns for the disjunction and the conditional:

(P	∨	~	Q)	·	(Q	⊃	P)
T	**T**	F	T	**T**	T	**T**	T
T	**T**	T	F	**T**	F	**T**	T
F	**F**	F	T	**F**	T	**F**	F
F	**T**	T	F	**T**	F	**T**	F

Thus, this proposition is false when P is false and Q is true, and true otherwise.
Note that you only have to write out the truth table once, like the last one in this demonstration.

Here is the start to an eight-line truth table, which we will complete on Monday:

[(P	⊃	Q)	·	(Q	⊃	R)]	⊃	(P	⊃	R)
T		T		T		T		T		T
T		T		T		F		T		F
T		F		F		T		T		T
T		F		F		F		T		F
F		T		T		T		F		T
F		T		T		F		F		F
F		F		F		T		F		T
F		F		F		F		F		F

Note that the columns under each instance of the same variable are identical.
In general, to construct a truth table:

> The first variable is assigned T in the top half and assigned F in the bottom half.
> The second variable is assigned T in the top quarter, F in the second quarter, T in the third quarter, and F in the bottom quarter.
> The third variable is assigned T in the top eight, F in the second eighth...
> ...
> The last variable is assigned alternating instances of T and F.

So, in an 128 row truth table (7 variables), the first variable would get 64 Ts and 64 Fs, the second variable would get 32 Ts, 32 Fs, 32 Ts, and 32 Fs, the third variable would alternate Ts and Fs in groups of 16, the fourth variable would alternate Ts and Fs in groups of 8s... and the seventh variable would alternate single instances of Ts and Fs.
It does not matter which variables we take as first, second, third, etc., but it is conventional that we work from left to right.
Remember that every instance of the same variable letter gets the same assignment of truth values.

VII. **Exercises C**. Construct truth tables for each of the following propositions.

1. ~P ⊃ Q
2. (P ≡ P) ⊃ P
3. ~Q ∨ (P ⊃ Q)

VIII. **Solutions**

Answers to Exercises A:
1. False
2. True
3. Unknown
4. True
5. False
6. True
7. True
8. True
9. False

Answers to Exercises B:
1. True
2. False
3. True

Answers to Exercises C

1.

~	P	⊃	Q
F	T	**T**	T
F	T	**T**	F
T	F	**T**	T
T	F	**F**	F

2.

(P	≡	P)	⊃	P
T	T	T	**T**	T
F	F	F	**F**	F

3.

~	Q	∨	(P	⊃	Q)
F	T	**T**	T	T	T
F	T	**T**	F	T	T
T	F	**T**	T	F	F
T	F	**T**	F	T	F